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Convex Structures and Effect Algebras

Stanley Gudder[1]

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Effect algebras have important applications in the foundations of quantum mechanics and in fuzzy probability theory. An effect algebra that possesses a convex structure is called a convex effect algebra. Our main result shows that any convex effect algebra admits a representation as a generating initial interval of an ordered linear space. This result is analogous to a classical representation theorem for convex structures due to M. H. Stone. We also give a relationship between a convex effect algebra and a statistical model called a convex effect-state space.

1. INTRODUCTION

An algebraic structure called an effect algebra has recently been introduced for investigations in the foundations of quantum mechanics [3, 13, 14]. Equivalent structures called D-posets and generalized orthoalgebras have also been studied [8, 10, 11, 15, 22, 23]. Moreover, effect algebras play a fundamental role in recent investigations of fuzzy probability theory [1, 2, 4, 5, 19]. In the quantum mechanical framework, the elements of an effect algebra P represent quantum effects and these are important for quantum statistics and quantum measurement theory [3, 6, 7]. One may think of a quantum effect as an elementary yes–no measurement that may be unsharp or imprecise. In the fuzzy probability setting, elements of P represent fuzzy events which are statistical events that may not be crisp or sharp. The quantum effects and fuzzy events are then used to construct general quantum measurements (or observables) and fuzzy random variables. The structure of an effect algebra is given by a partially defined binary operation $\oplus$ that is used to form a combination $a \oplus b$ of effects $a, b \in P$. The element $a \oplus b$ represents

[1] Department of Mathematics and Computer Science, University of Denver, Denver, Colorado 80208.

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a statistical combination of a and b whose probability of occurrence equals the sum of the probabilities that a and b occur individually.

The common examples of effect algebras that are employed in practice also possess a convex structure. For example, if a is a quantum effect and $\lambda \in [0, 1]$, then λa represents the effect a attentuated by a factor of λ. A similar interpretation is given for fuzzy events. Then $\lambda a \oplus (1 - \lambda)b$ is a generalized convex combination that can be constructed in practice. Due to the operational significance of such combinations it seems desirable to investigate effect algebras that possess an additional convex structure and we call them convex effect algebras.

General convex structures have important applications to studies in color vision, decision theory, operational quantum mechanics, and economics [12, 16, 17, 27, 28]. A classical representation theorem of M. H. Stone [16, 26] has sometimes been useful in these studies. This theorem states that certain convex structures can be represented as convex subsets of a real linear space. In this paper, we present an analogous theorem for convex effect algebras. Although there are some similarities between our proof and that of Stone, a much more delicate argument must be used because we have to preserve the effect algebra structure as well as the convex structure. Also, since our structure is richer than a convex structure alone, we obtain a stronger theorem. In Stone's theorem, a convex structure is represented by a convex base of a positive cone K that generates an ordered linear space (V, K). Our theorem states that a convex effect algebra can be represented by an initial interval $[\theta, u]$ that generates an ordered linear space (V, K). An interval $[\theta, u]$ in (V, K) has a natural effect algebra structure and we call $[\theta, u]$ a linear effect algebra. A linear effect algebra is a special case of an interval effect algebra which has recently been investigated [14]. In this note, we shall only state the representation theorem and the proof will appear elsewhere [20]. We shall also give a relationship between a convex effect algebra and a statistical model called a convex effect-state space.

2. DEFINITIONS AND BASIC RESULTS

An *effect algebra* is an algebraic system $(P, 0, 1, \oplus)$ where 0, 1 are distinct elements of P and $\oplus$ is a partial binary operation on P that satisfies the following conditions.

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.

(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(E3) For every $a \in P$ there exists a unique $a' \in P$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$.

(E4) If $a \oplus 1$ is defined, then $a = 0$.

We define $a \leq b$ if there exists a $c \in P$ such that $a \oplus c = b$. It can be shown that $(P, 0, 1, \leq)$ is a bounded poset and $a \oplus b$ is defined if and only if $a \leq b'$ [11, 13]. If $a \leq b'$, we write $a \perp b$. An important property of an effect algebra is the *cancellation law*, which states that a $a \oplus b = a \oplus c$ implies $b = c$. Moreover, it can be shown that $a'' = a$ and that $a \leq b$ implies $b' \leq a'$ for every $a, b \in P$ [11, 13].

An effect algebra P is *convex* if for every $a \in P$ and $\lambda \in [0, 1] \subseteq \mathbb{R}$ there exists an element $\lambda a \in P$ such that the following conditions hold.

(C1) If $\alpha, \beta \in [0, 1]$ and $a \in P$, then $\alpha(\beta a) = (\alpha\beta)a$.

(C2) If $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ and $a \in P$, then $\alpha a \perp \beta a$ and $(\alpha + \beta)a = \alpha a \oplus \beta a$.

(C3) If $a, b \in P$ with $a \perp b$ and $\lambda \in [0, 1]$, then $\lambda a \perp \lambda b$ and $\lambda(a \oplus b) = \lambda a \oplus \lambda b$.

(C4) If $a \in P$, then $1a = a$.

A map $(\lambda, a) \mapsto \lambda a$ that satisfies (C1)–(C4) is an example of a bimorphism from $[0, 1] \times P$ into P [10] and we call this map a *convex structure* on P. Notice that $0a = 0$ for every $a \in P$. Indeed, by (C2) and (C4) we have

$$0a \oplus a = (0 + 1)a = 1a = a = 0 \oplus a$$

so by the cancellation law $0a = 0$.

The effect algebras that arise in practice are usually convex. For example, let H be a complex Hilbert space and let $\mathscr{E}(H)$ be the set of operators on H that satisfy $0 \leq A \leq I$, where we are using the usual ordering of bounded operators. For $A, B \in \mathscr{E}(H)$, we write $A \perp B$ if $A + B \in \mathscr{E}(H)$ and in this case we define $A \oplus B = A + B$. It is clear that $(\mathscr{E}(H), 0, I, \oplus)$ is an effect algebra and we call $\mathscr{E}(H)$ a *Hilbert space effect algebra*. Hilbert space effect algebras are important in foundational studies of quantum mechanics [6, 7, 9, 21, 24, 25]. For $\lambda \in [0, 1]$ and $A \in \mathscr{E}(H)$, λA is the usual scalar multiplication for operators. This gives a convex structure on $\mathscr{E}(H)$, so $\mathscr{E}(H)$ becomes a convex effect algebra. For another example, let $(\Omega, \mathscr{A})$ be a measurable space and let $\mathscr{E}(\Omega, \mathscr{A})$ be the set of measurable functions on Ω with values in $[0, 1]$. If we define $\oplus$ and scalar multiplication λf analogously as in the previous example, we see that $\mathscr{E}(\Omega, \mathscr{A})$ is a convex effect algebra. The elements of $\mathscr{E}(\Omega, \mathscr{A})$ are called *fuzzy events* and they are the basic concepts in fuzzy probability theory [1, 2, 4, 5, 19].

We now consider a more general type of convex effect algebra called a linear effect algebra. It is no accident that the previous two examples are linear effect algebras because we shall show that any convex effect algebra is equivalent to a linear effect algebra. A linear effect algebra is an initial interval in the positive cone of an ordered linear space. We now give the precise definitions.

Let V be a real linear space with zero θ. A subset K of V is a *positive cone* if $\mathbb{R}^+ K \subseteq K$, $K + K \subseteq K$, and $K \cap (-K) = \{\theta\}$. For $x, y \in V$ we define $x \leq_K y$ if $y - x \in K$. Then $\leq_K$ is a partial order on V and we call (V, K) an *ordered linear space* with positive cone K. We say that K is *generating* if $V = K - K$. Let $u \in K$ with $u \neq \theta$ and form the interval

$$[\theta, u] = \{x \in K: x \leq_K u\}$$

For $x, y \in [\theta, u]$ we write $x \perp y$ if $x + y \leq_K u$ and in this case we define $x \oplus y = x + y$. It is clear that $([\theta, u], \theta, u, \oplus)$ is an effect algebra with $x' = u - x$ for every $x \in [\theta, u]$. This is an example of an interval effect algebra [14]. It is also easy to check that $[\theta, u]$ is a convex subset of K. It follows that if $\lambda \in [0, 1]$ and $x \in [\theta, u]$, then

$$\lambda x = \lambda x + (1 - \lambda)\theta \in [\theta, u]$$

A straightforward verification shows that $(\lambda, x) \mapsto \lambda x$ is a convex structure on $[\theta, u]$ so that $[\theta, u]$ is a convex effect algebra which we call a *linear effect algebra*. We say that $[\theta, u]$ *generates* K if $K = \mathbb{R}^+ [\theta, u]$ and we say that $[\theta, u]$ *generates* V if $[\theta, u]$ generates K and K generates V. A simple example of a linear effect algebra is the unit interval $[0, 1]$ in the linear space $\mathbb{R}$ with its usual order. Of course, $[0, 1]$ generates $\mathbb{R}$. Two ordered linear spaces (V_1, K_1) and (V_2, K_2) are *order isomorphic* if there exists a linear bijection $T: V_1 \to V_2$ such that $T(K_1) = K_2$.

Because of the associative law (E2), we do not have to write parentheses for orthogonal sums of three or more elements. If a is an element of an effect algebra and $a \oplus a \oplus \cdots \oplus a$ is defined (n summands), then we denote this element by na. Our first result summarizes some basic properties of a convex effect algebra [20].

Lemma 2.1. Let P be a convex effect algebra. (i) If $a \leq b$, then $\lambda a \leq \lambda b$ for every $\lambda \in [0, 1]$. (ii) If $0 \leq \alpha \leq \beta \leq 1$, then $\alpha a \leq \beta a$ for every $a \in P$. (iii) If $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, then $\alpha a \perp \beta b$ for every $a, b \in P$. (iv) For $\lambda \in (0, 1)$, $\lambda a = 0$ if and only if $a = 0$. (v) If na is defined for $n \in \mathbb{N}$ and $0 \leq \lambda \leq 1/n$, then $\lambda(na) = (\lambda n)a$. (vi) If na is defined for $n \in \mathbb{N}$ and $\lambda \in [0, 1]$, then $n(\lambda a)$ is defined and $n(\lambda a) = \lambda(na)$. (vii) If $\lambda \in (0, 1]$ and $\lambda a = \lambda b$, then $a = b$. (viii) If $a \neq 0$, $\alpha, \beta \in [0, 1]$, and $\alpha a = \beta a$, then $\alpha = \beta$.

It follows from Lemma 2.1(iii) that a convex effect algebra P is "convex" in the following sense. If $\lambda \in [0, 1]$ and $a, b \in P$, then $\lambda a \oplus (1 - \lambda)b$ is defined and hence is an element of P.

If P and Q are effect algebras, a map $\phi: P \to Q$ is *additive* if $a \perp b$ implies that $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. An additive map ϕ that satisfies $\phi(1) = 1$ is called a *morphism*. A morphism $\phi: P \to Q$ for which $\phi(a) \perp \phi(b)$ implies that $a \perp b$ is called a *monomorphism*. A surjective monomorphism is called an *isomorphism*. It is easy to show that if ϕ is an isomorphism, then ϕ is injective and ϕ^{-1} is an isomorphism. If P and Q are convex effect algebras, a morphism $\phi: P \to Q$ is called an *affine morphism* if $\phi(\lambda a) = \lambda\phi(a)$ for every $\lambda \in [0, 1]$, $a \in P$. It follows from Lemma 2.1(iii) that an affine morphism preserves convex combinations in the sense that if $\lambda \in [0, 1]$ and $a, b \in P$, then

$$\phi(\lambda a \oplus (1 - \lambda)b) = \lambda\phi(a) \oplus (1 - \lambda)\phi(b)$$

An isomorphism $\phi: P \to Q$ that is affine is called an *affine isomorphism* and if such a ϕ exists, we say that P and Q are *affinely isomorphic*. Notice that if $\phi: P \to Q$ is an affine isomorphism, then $\phi^{-1}: Q \to P$ is also an affine isomorphism. Indeed, let $\lambda \in [0, 1]$ and $b \in Q$. Then there exists an $a \in P$ such that $\phi(a) = b$ so that $\phi(\lambda a) = \lambda b$. Hence,

$$\phi^{-1}(\lambda b) = \lambda a = \lambda\phi^{-1}(b)$$

Lemma 2.2. If P is a convex effect algebra, Q is an effect algebra, and $\phi: P \to Q$ is an isomorphism, then there exists a unique convex structure on Q such that ϕ is an affine isomorphism.

3. REPRESENTATION THEOREM

We now present a representation theorem for convex effect algebras [20]. This theorem is analogous to a representation theorem for convex structures due to M. H. Stone [16, 26].

Theorem 3.1. If $(P, 0, 1, \oplus)$ is a convex effect algebra, then P is affinely isomorphic to a linear effect algebra $[\theta, u]$ that generates an ordered linear space (V, K) and the effect algebra order $\leq$ on $[\theta, u]$ coincides with linear space order $\leq_K$ restricted to $[\theta, u]$. Moreover, (V, K) is unique in the sense that if P is affinely isomorphic to a linear effect algebra $[\theta_1, u_1]$ that generates (V_1, K_1), then (V_1, K_1) is order isomorphic to (V, K).

If P is an effect algebra, a morphism $\omega: P \to [0, 1] \subseteq \mathbb{R}$ is called a *state*. We denote the set of states on P by $\Omega(P)$. The states correspond to initial conditions or preparations of a system and $\omega(a)$ is interrupted as the probability that the effect a occurs when the system is in the state ω.

Lemma 3.2. If P is a convex effect algebra, then every $\omega \in \Omega(P)$ is affine.

Proof. Let R be the set of rationals in $[0, 1]$. If $n \in \mathbb{N}$, then we have

$$\omega(a) = \omega\left(\frac{1}{n}a \oplus \cdots \oplus \frac{1}{n}a\right) = n\omega\left(\frac{1}{n}a\right) \qquad (n \text{ summands})$$

so $\omega(1/n\ a) = 1/n\ \omega(a)$. If $m, n \in \mathbb{N}$, $m \leq n$, then

$$\omega\left(\frac{m}{n}a\right) = \omega\left(\frac{1}{n}a \oplus \cdots \oplus \frac{1}{n}a\right) = m\omega\left(\frac{1}{n}a\right) = \frac{m}{n}\omega(a) \qquad (m \text{ summands})$$

Hence, $\omega(ra) = r\omega(a)$ for every $r \in R$. If $\lambda \in [0, 1]$, then

$$\begin{aligned}\omega(\lambda a) &\leq \inf\{\omega(ra): r \in R, r \geq \lambda\} = \omega(a) \inf\{r \in R: r \geq \lambda\} \\ &= \lambda\omega(a)\end{aligned}$$

Similarly, $\omega(\lambda a) \geq \lambda\omega(a)$ so the result follows. ■

There are examples of effect algebras for which $\Omega(P) = \emptyset$. It can be shown that if P is a convex effect algebra, then $\Omega(P) \neq \emptyset$. However, even if P is convex, $\Omega(P)$ may contain only one element and it is important in applications to have a rich supply of states. We say that $\Omega(P)$ is *order determining* if $\omega(a) \leq \omega(b)$ for all $\omega \in \Omega(P)$ implies that $a \leq b$. We now give a condition on a convex effect algebra P that ensures an order-determining set of states. We say that P is *archimedean* if whenever $a, b, c \in P$ with $a \perp b$ and $c \leq a \oplus 1/n\ b$ for every $n \in \mathbb{N}$, then $c \leq a$. It is easy to see that if $\Omega(P)$ is order determining, then P is archimedean. Indeed, suppose that $c \leq a \oplus 1/n\ b$ for every $n \in \mathbb{N}$. Then for every $\omega \in \Omega(P)$, $n \in \mathbb{N}$, we have $\omega(c) \leq \omega(a) + 1/n\ \omega(b)$. Hence, $\omega(c) \leq \omega(a)$ for $\omega \in \Omega(P)$, so that $c \leq a$. The following result, which relies on Theorem 3.1 and Lemma 3.2, shows that the converse holds.

Theorem 3.3. If P is a convex effect algebra, then $\Omega(P)$ is order determining if and only if P is archimedean.

4. EFFECT-STATE SPACES

In the previous sections we assumed that the effects for a physical system formed an effect algebra P and that the states were morphisms from P into $[0, 1] \subseteq \mathbb{R}$. We now treat the effects and states as undefined concepts and derive their properties from a few simple, natural axioms.

Most statistical theories for physical systems contain two basic primitive concepts, namely effects and states. The effects correspond to simple yes–no measurements or experiments and the states correspond to preparation procedures that specify the initial conditions of the system being measured. For example, a particle detector with domain of sensitivity $\Delta \subseteq \mathbb{R}^3$ corresponds to an effect. If a particle of a certain type is in Δ at a certain time, then the detector registers (yes) and otherwise it does not register (no). This effect either occurs (is observed) or does not occur (is not observed), so the measurement has only two values, yes or no. We also include the possibility that the detector is not perfectly accurate. The state of the system prescribes the initial conditions of the tested particles; for example, their initial position, momentum, energy, or spin. Usually, each effect a and state s experimentally determine a probability $F(a, s)$ that the effect a occurs when the system has been prepared in the state s. For a given physical system, denote its set of possible effects by $\mathscr{E}$ and its set of possible states by $\mathscr{S}$. In a reasonable statistical theory, the probability function F satisfies two axioms that are given in the following definition.

An *effect-state space* is a triple $(\mathscr{E}, \mathscr{S}, F)$ where $\mathscr{E}$ and $\mathscr{S}$ are nonempty sets and F is a mapping from $\mathscr{E} \times \mathscr{S}$ into $[0, 1] \subseteq \mathbb{R}$ satisfying:

(ES1) there exist elements $0, 1 \in \mathscr{E}$ such that $F(0, s) = 0$, $F(1, s) = 1$ for every $s \in \mathscr{S}$.

(ES2) if $F(a, s) \leq F(b, s)$ for every $s \in \mathscr{S}$, then there exists a unique $c \in \mathscr{E}$ such that $F(a, s) + F(c, s) = F(b, s)$ for every $s \in \mathscr{S}$.

The elements 0, 1 in (ES1) correspond to the null effect that never occurs and the unit effect that always occurs, respectively. Condition (ES2) postulates that if a has a smaller probability of occurring than b in every state, then there exists a unique effect c which when combined with a gives the probability that b occurs in every state. The next lemma is proved in ref. 18.

Lemma 4.1. Let $(\mathscr{E}, \mathscr{S}, F)$ be an effect-state space. If $F(a, s) + F(b, s) \leq 1$ for every $s \in \mathscr{S}$, then there exists a unique $c \in \mathscr{E}$ such that $F(c, s) = F(a, s) + F(b, s)$ for every $s \in \mathscr{S}$.

For an effect-state space $(\mathscr{E}, \mathscr{S}, F)$ we write $a \perp b$ if $F(a, s) + F(b, s) \leq 1$ for every $s \in \mathscr{S}$ and the unique c in Lemma 4.1 is denoted $c = a \oplus b$. The following theorem, which is proved in ref. 18, shows that effect algebras arise naturally from effect-state spaces.

Theorem 4.2. If $(\mathscr{E}\ \mathscr{S}, F)$ is an effect-state space and $S = \{F(\cdot, s): s \in \mathscr{S}\}$, then $(\mathscr{E}, 0, 1, \oplus)$ is an effect algebra with an order-determining set of states S. Conversely, if $(P, 0, 1, \oplus)$ is an effect algebra and S is an order-determining set of states on P, then (P, S, F) is an effect-state space where $F: P \times S \to [0, 1]$ is defined by $F(a, s) = s(a)$.

We now show that a natural convex structure can be defined for an effect-state space $(\mathscr{E}, \mathscr{S}, F)$. We say that $(\mathscr{E}, \mathscr{S}, F)$ is a *convex effect-state space* if for every $a \in \mathscr{E}$ and $\lambda \in [0, 1] \subseteq \mathbb{R}$ there exists an element $\lambda a \in \mathscr{E}$ such that $F(\lambda a, s) = \lambda F(a, s)$ for every $s \in \mathscr{S}$. The unique element $\lambda a \in \mathscr{E}$ is interpreted as the effect a attenuated by the factor λ. If $(P, 0, 1, \oplus)$ is a convex effect algebra and S is an order-determining set of states on P, then it follows from Theorem 4.2 and Lemma 3.2 that (P, S, F) is a convex effect-state space where $F: P \times S \to [0, 1]$ is defined by $F(a, s) = s(a)$. The next result shows that the converse holds.

Theorem 4.3. If $(\mathscr{E}, \mathscr{S}, F)$ is a convex effect-state space and $S = \{F(\cdot, s): s \in \mathscr{S}\}$, then $(\mathscr{E}, 0, 1, \oplus)$ is a convex effect algebra with an order-determining set of states S.

Proof. It follows from Theorem 4.2 that $(\mathscr{E}, 0, 1, \oplus)$ is an effect algebra with an order-determining set of states S. It only remains to show that $\mathscr{E}$ is convex; that is, properties (C1)–(C4) hold. To verify (C1) we have for every $s \in \mathscr{S}$

$$F(\alpha(\beta a), s) = \alpha F(\beta a, s) = (\alpha\beta)F(a, s) = F((\alpha\beta)a, s))$$

Hence, $\alpha(\beta a) = (\alpha\beta)a$. To verify (C2), suppose that $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then for every $s \in \mathscr{S}$ we have

$$F(\alpha a, s) + F(\beta a, s) = \alpha F(a, s) + \beta F(a, s) = (\alpha + \beta)F(a, s)$$
$$= F((\alpha + \beta)a, s) \leq 1$$

Hence, $\alpha a \perp \beta a$ and $(\alpha + \beta)a = \alpha a \oplus \beta a$. To verify (C3), suppose that $a \perp b$ and $\lambda \in [0, 1]$. Then for every $s \in \mathscr{S}$ we have

$$F(\lambda a, s) + F(\lambda b, s) = \lambda[F(a, s) + F(b, s)] = \lambda F(a \oplus b, s)$$
$$= F(\lambda(a \oplus b), s)$$

Hence, $\lambda a \perp \lambda b$ and $\lambda(a \oplus b) = \lambda a \oplus \lambda b$. Finally, (C4) holds because $F(1a, s) = F(a, s)$ for every $s \in \mathscr{S}$. ■

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